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Mean-variance inefficiency of CRRA and CARA utility functions for portfolio selection in defined contribution pension schemes

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Abstract

We consider the portfolio selection problem in the accumulation phase of a defined contribution pension scheme in continuous time, and compare the mean-variance and the expected utility maximization approaches. Using the embedding technique pioneered by Zhou and Li (2000) we first find the efficient frontier of portfolios in the Black-Scholes financial market. Then, using standard stochastic optimal control we find the optimal portfolios derived via expected utility for popular utility functions. As a main result, we prove that the optimal portfolios derived with the CARA and CRRA utility functions are not mean-variance efficient. As a corollary, we prove that this holds also in the standard portfolio selection problem. We provide a natural measure of inefficiency based on the difference between optimal portfolio variance and minimal variance, and we show its dependence on risk aversion, Sharpe ratio of the risky asset, time horizon, initial wealth and contribution rate. Numerical examples illustrate the extent of inefficiency of CARA and CRRA utility functions in defined contribution pension schemes.

Keywords. Mean-variance approach, efficient frontier, expected utility maximization, defined contribution pension scheme, portfolio selection, risk aversion, Sharpe ratio.

JEL classification: C61, D81, G11, G23.

1 Introduction

1.1 The problem

The crisis of international Pay As You Go public pension systems is forcing governments of most countries to drastically cut pension benefits of future generations and to encourage the development of fully funded pension schemes. It is well-known that the reforms undertaken in most industrialized countries give a preference towards defined contribution (DC) plans rather than defined benefit (DB) plans. Thus, defined contribution pension schemes will play a crucial role in the social pension systems and financial advisors of DC plans will be needing flexible decision making tools to be appropriately tailored to a member's needs, to help her making optimal and conscious choices. Given that the member of a defined contribution pension scheme has some freedom in choosing the

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investment allocation of her fund in the accumulation phase, she has to solve a portfolio selection problem. Traditionally, the usual way to deal with it has been maximization of expected utility of final wealth. In this paper, we show that the optimal portfolios derived with CARA and CRRA utility functions are not mean-variance efficient. In fact, we prove that the variance of the optimal portfolios is not the minimal variance. As a byproduct, we propose a natural measure of inefficiency, based on the difference between optimal portfolio variance and minimal variance. We show that in general inefficiency increases with time horizon and Sharpe ratio of the risky asset and decreases with risk aversion. We also prove that the amount of wealth invested in the risky asset at any time is strictly positive. This means that short-selling of the risky asset is prevented by the adoption of mean-variance efficient strategies in continuous-time. This result, that is a desirable feature for pension funds management, is standard in the single-period framework and is proven to hold true also in the continuous-time setting.

1.2 Review of the literature

The literature on the accumulation phase of defined contribution pension schemes is full of examples of optimal investment strategies resulting from expected utility maximization. See for instance Battocchio and Menoncin (2004), Boulrier, Huang and Taillard (2001), Cairns, Blake and Dowd (2006), Deelstra, Grasselli and Koehl (2003), Devolder, Bosch Princep and Dominguez Fabian (2003), Di Giacinto, Gozzi and Federico (2009b), Gao (2008), Haberman and Vigna (2002), Xiao, Zhai and Qin (2007). Consistently with the economics and financial literature, the most widely used utility function exhibits constant relative risk aversion (CRRA); i.e., the power or logarithmic utility function, see, e.g., Boulrier et al. (2001), Cairns et al. (2006), Deelstra et al. (2003), Devolder et al. (2003), Gao (2008), Xiao et al. (2007). Some papers use the utility function that exhibits constant absolute risk aversion (CARA); i.e., the exponential utility function, see, e.g., Battocchio and Menoncin (2004), Devolder et al. (2003). Finally, Di Giacinto et al. (2009b) use a general form of utility function that includes as a special case a modified version of the power utility function, and Haberman and Vigna (2002) minimize expected loss using a quadratic loss function, a common approach in pension schemes optimization.

In the context of DC pension funds the problem of finding the optimal investment strategy that is mean-variance efficient, i.e. minimizes the variance of the final fund given a certain level of expected value of the fund has not been reported in published articles. This is not surprising and is mainly due to the fact that the exact and rigorous multi-period and continuous-time versions of the mean-variance problem have been produced only quite recently. The main reason of this delay in solving such a relevant problem, since Markowitz (1952) and Markowitz (1959), lies in the difficulty inherent in the extension from single-period to multi-period or continuous-time framework. In the portfolio selection literature the problem of finding the minimum variance trading strategy in continuous-time has been solved by Richardson (1989) via the martingale approach. The same approach has been used also by Bajeux-Besnainou and Portait (1998) in a more general framework. They also find the dynamic efficient frontier and compare it to the static single-period one. Regarding the use of stochastic control theory to solve a mean-variance optimization problem, a real breakthrough was introduced by Li and Ng (2000) in a discrete-time multi-period framework and Zhou and Li (2000) in a continuous-time model. They show how to transform the difficult problem into a tractable one, by embedding the original problem into a stochastic linear-quadratic control problem, that can then be solved through standard methods. These seminal papers have been followed by a number of extensions; see, for instance, Bielecky, Jin, Pliska and Zhou (2005) and references therein. In the context of DC pension schemes, the techniques of Zhou and Li (2000) have been used by Højgaard and Vigna (2007). They find the efficient frontier for the two-asset case as well as for the $n+1$

asset case and show that the target-based approach, based on the minimization of a quadratic loss function, can be formulated as a mean-variance optimization problem, which is an expected result.

Mean-variance and expected utility are two different approaches for dealing with portfolio selection. It is well-known that in the single-period framework the mean-variance approach and expected utility optimization coincide if either the utility function is quadratic or the asset returns are normal. Furthermore, in the continuous-time framework when prices are log-normal there is consistency between optimal choices and mean-variance efficiency at instantaneous level (see Merton (1971) and also Campbell and Viceira (2002)). However, this does not imply that an optimal policy should *remain* efficient also after two consecutive instants or, more in general, on a time interval greater than the instantaneous one. In fact, in general it does not. In previous financial literature, the lack of efficiency of optimal policies in continuous-time was noted for instance by some empirical works that compare mean-variance efficient portfolios with expected utility optimal portfolios and find that there are indeed differences between those portfolios. Among these, Hakansson (1971), Grauer (1981) and Grauer and Hakansson (1993). Related work can be found in Zhou (2003).

The aim of this paper is to compare the two leading alternatives for portfolio selection in DC pension schemes, i.e. expected utility and mean-variance optimization. Although the fact that a myopically efficient policy is not necessarily efficient over the entire period is not a new result and has been already mentioned (see e.g. Bajeux-Besnainou and Portait (1998)), this paper proves it directly and focuses on the more interesting aspect of extent of inefficiency whenever popular utility functions are used. In particular, we prove inefficiency of final portfolios on a given time period $(0, T)$ when CRRA and CARA utility functions are maximized. We introduce a natural measure of inefficiency of optimal portfolio derived with those utility functions, and find the intuitive results that in general inefficiency increases with time horizon T and Sharpe ratio of the risky asset and decreases with risk aversion. This drawback becomes particularly relevant in applications to pension funds, given the long-term horizon involved and the fact that investors should care more about behaving efficiently on the entire time horizon rather than in each single instant. Finally, we prove that the amount invested in the risky asset with a mean-variance efficient strategy is strictly positive. In other words, short-selling is prevented a priori with adoption of efficient strategies. This result, together with the recent result by Chiu and Zhou (2009) that an efficient portfolio must have a non-zero but not necessarily positive allocation to the riskless asset, sheds further light on the composition of efficient portfolios. We end with a numerical example, aimed at showing, in the context of a DC pension scheme, the extent of inefficiency of optimal portfolios derived with CRRA and CARA utility functions with typical risk aversion coefficients.

1.3 Agenda of the paper

The remainder of the paper is organized as follows. To improve readability of the paper, almost all proofs and derivations of intermediate results are relegated in the Appendix. In section 2, we introduce the model. In section 3, we report the mean-variance optimization problem solved by Højgaard and Vigna (2007). In section 4, we outline the expected utility optimization approach and solve the optimization problem with the CARA and CRRA utility functions. In section 5, we state and prove a theorem that shows that the optimal portfolios derived in section 4 are not efficient in the mean-variance setting. In section 6, we define a measure of inefficiency, based on the difference between the variance of the CARA and CRRA optimal inefficient portfolios and the corresponding minimal variance and analyze the dependence of inefficiency on risk aversion parameter, Sharpe ratio of risky asset, time horizon, initial wealth and contribution rate. In section 7, we show that the target-based approach is mean-variance efficient. In section 8, we report a numerical example,

aimed at showing the extent of inefficiency by adopting popular utility functions in a DC pension plan. Section 9 concludes and outlines further research.

2 The model

A member of a defined contribution pension scheme is faced with the problem of how to invest optimally the fund at her disposal and the future contributions to be paid in the fund. The financial market available for her portfolio allocation problem is the Black-Scholes model (see e.g. Björk (1998)). This consists of two assets, a riskless one, whose price $B(t)$ follows the dynamics:

$$dB(t) = rB(t)dt, \quad (1)$$

where $r > 0$, and a risky asset, whose price dynamics $S(t)$ follows a geometric Brownian motion with drift $\lambda > 0$ and diffusion $\sigma > 0$:

$$dS(t) = \lambda S(t)dt + \sigma S(t)dW(t), \quad (2)$$

where $W(t)$ is a standard Brownian motion defined on a complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbf{P})$, with $\mathcal{F}_t = \sigma\{W(s) : s \leq t\}$.

The constant contribution rate paid in the unit time in the fund is $c \geq 0$. The proportion of portfolio invested in the risky asset at time t is denoted by $y(t)$. The fund at time t , $X(t)$, grows according to the following SDE:

$$\begin{aligned} dX(t) &= \{X(t)[y(t)(\lambda - r) + r] + c\}dt + X(t)y(t)\sigma dW(t) \\ X(0) &= x_0 \geq 0. \end{aligned} \quad (3)$$

The amount x_0 is the initial fund paid in the member's account, which can also be null, if the member has just joined the scheme with no transfer value from another fund. The member enters the plan at time 0 and contributes for T years, after which she retires and withdraws all the money (or converts it into annuity). The temporal horizon T is supposed to be fixed, e.g. T can be 20, 30 years, depending on the member's age at entry.

The member of the pension plan has to choose the criterion for her portfolio selection problem. She restricts her attention to the two leading approaches, mean-variance approach and expected utility maximization.

3 The mean-variance approach

In this section, we assume that the individual chooses the mean-variance approach for her portfolio selection problem. She then pursues the two conflicting objectives of maximum expected final wealth together with minimum variance of final wealth, namely she seeks to minimize the vector

$$[-E(X(T)), \text{Var}(X(T))].$$

Definition 1 *An investment strategy $y(\cdot)$ is said to be admissible if $y(\cdot) \in L^2_{\mathcal{F}}(0, T; \mathbf{R})$.*

Definition 2 The mean-variance optimization problem is defined as

$$\begin{aligned} & \text{Minimize} \quad (J_1(y(\cdot)), J_2(y(\cdot))) \equiv (-E(X(T)), \text{Var}(X(T))) \\ & \text{subject to} \quad \begin{cases} y(\cdot) & \text{admissible} \\ X(\cdot), y(\cdot) & \text{satisfy (3)}. \end{cases} \end{aligned} \quad (4)$$

An admissible strategy $\bar{y}(\cdot)$ is called an efficient strategy if there exists no admissible strategy $y(\cdot)$ such that

$$J_1(y(\cdot)) \leq J_1(\bar{y}(\cdot)) \quad J_2(y(\cdot)) \leq J_2(\bar{y}(\cdot)), \quad (5)$$

and at least one of the inequalities holds strictly. In this case, the point $(J_1(\bar{y}(\cdot)), J_2(\bar{y}(\cdot))) \in \mathbf{R}^2$ is called an efficient point and the set of all efficient points is called the efficient frontier.

Problem (4) is equivalent to

$$\min_{y(\cdot)} [-E(X(T)) + \alpha \text{Var}(X(T))], \quad (6)$$

where $\alpha > 0$. Notice that α is a measure of the risk aversion of the individual. Zhou and Li (2000) show that problem (6) is equivalent to

$$\begin{aligned} & \text{Minimize} \quad (J(y(\cdot)), \alpha, \mu) \equiv E[\alpha X(T)^2 - \mu X(T)], \\ & \text{subject to} \quad \begin{cases} y(\cdot) & \text{admissible} \\ X(\cdot), y(\cdot) & \text{satisfy (3)}. \end{cases} \end{aligned} \quad (7)$$

where

$$\mu = 1 + 2\alpha E(\bar{X}(T)). \quad (8)$$

In solving problem (7) we follow the approach presented in Zhou and Li (2000). The derivation of the solution of this LQ control problem is fully contained in Højgaard and Vigna (2007), so we refer the interested reader to this paper for details and here report only the solution. The optimal investment allocation at time t , given that the fund is x , is given by

$$\bar{y}(t, x) = -\frac{\lambda - r}{\sigma^2 x} \left[x - \delta e^{-r(T-t)} + \frac{c}{r}(1 - e^{-r(T-t)}) \right], \quad (9)$$

where

$$\delta = \frac{\mu}{2\alpha}. \quad (10)$$

The evolution of the fund under optimal control $\bar{X}(t)$ is easily obtained:

$$\begin{aligned} d\bar{X}(t) = & \left[(r - \beta^2)\bar{X}(t) + e^{-r(T-t)}(\beta^2\delta + \frac{\beta^2 c}{r}) + (c - \frac{\beta^2 c}{r}) \right] dt + \\ & + \left[-\beta\bar{X}(t) + e^{-r(T-t)}(\beta\delta + \frac{\beta c}{r}) - \frac{\beta c}{r} \right] dW(t), \end{aligned} \quad (11)$$

where

$$\beta := \frac{\lambda - r}{\sigma}, \quad (12)$$

is the Sharpe ratio of the risky asset. By application of Ito's lemma to (11), we obtain the SDE that governs the evolution of $\bar{X}^2(t)$:

$$\begin{aligned} d\bar{X}^2(t) = & [(2r - \beta^2)\bar{X}^2(t) + 2c\bar{X}(t) + \beta^2((\delta + \frac{c}{r})e^{-r(T-t)} - \frac{c}{r})^2]dt + \\ & - 2\beta\{\bar{X}^2(t) - [(\delta + \frac{c}{r})e^{-r(T-t)} - \frac{c}{r}]\bar{X}(t) + \frac{c}{r}\}dW(t). \end{aligned} \quad (13)$$

If we take expectations on both sides of (11) and (13), we find that the expected value of the optimal fund and the expected value of its square follow the linear ODE's:

$$\begin{aligned} dE(\bar{X}(t)) &= [(r - \beta^2)E(\bar{X}(t)) + e^{-r(T-t)}\beta^2(\delta + \frac{c}{r}) + (c - \frac{\beta^2 c}{r})]dt \\ E(\bar{X}(0)) &= x_0 \end{aligned} \quad (14)$$

and

$$\begin{aligned} dE(\bar{X}^2(t)) &= [(2r - \beta^2)E(\bar{X}^2(t)) + 2cE(\bar{X}(t)) + \beta^2((\delta + \frac{c}{r})e^{-r(T-t)} - \frac{c}{r})^2]dt \\ E(\bar{X}^2(0)) &= x_0^2. \end{aligned} \quad (15)$$

By solving the ODE's we find that the expected value of the fund under optimal control at time t is

$$E(\bar{X}(t)) = \left(x_0 + \frac{c}{r}\right)e^{-(\beta^2-r)t} + \left(\delta + \frac{c}{r}\right)e^{-r(T-t)} - \left(\delta + \frac{c}{r}\right)e^{-r(T-t)-\beta^2 t} - \frac{c}{r}, \quad (16)$$

and the expected value of the square of the fund under optimal control at time t is:

$$\begin{aligned} E(\bar{X}^2(t)) &= \left(x_0 + \frac{c}{r}\right)^2 e^{-(\beta^2-2r)t} - \left(\delta + \frac{c}{r}\right)^2 e^{-2r(T-t)-\beta^2 t} - \frac{2c}{r}\left(\delta + \frac{c}{r}\right)e^{-r(T-t)} \\ &+ \frac{2c}{r}\left(\delta + \frac{c}{r}\right)e^{-r(T-t)-\beta^2 t} - \frac{2c}{r}\left(x_0 + \frac{c}{r}\right)e^{-(\beta^2-r)t} + \left(\delta + \frac{c}{r}\right)^2 e^{-2r(T-t)} + \frac{c^2}{r^2}. \end{aligned} \quad (17)$$

At terminal time T we have:

$$E(\bar{X}(T)) = \left(x_0 + \frac{c}{r}\right)e^{-(\beta^2-r)T} + \delta(1 - e^{-\beta^2 T}) - \frac{c}{r}e^{-\beta^2 T}, \quad (18)$$

and

$$E(\bar{X}^2(T)) = \left(x_0 + \frac{c}{r}\right)^2 e^{-(\beta^2-2r)T} + \delta^2(1 - e^{-\beta^2 T}) - \frac{2c}{r}\left(x_0 + \frac{c}{r}\right)e^{-(\beta^2-r)T} + \frac{c^2}{r^2}e^{-\beta^2 T}. \quad (19)$$

We now define an important quantity, that will play a special role in the rest of the paper:

$$\bar{x}_0 := x_0 e^{rT} + \frac{c}{r}(e^{rT} - 1). \quad (20)$$

The meaning of \bar{x}_0 is clear: it is the fund that would be available at time T investing initial fund and contributions in the riskless asset. The expected optimal final fund can be rewritten in terms of α, β and \bar{x}_0 :

$$E(\bar{X}(T)) = \bar{x}_0 + \frac{e^{\beta^2 T} - 1}{2\alpha} \quad (21)$$

It is easy to see that the expected optimal final fund is the sum of the fund that one would get investing the whole portfolio always in the riskless asset plus a term, $\frac{e^{\beta^2 T}-1}{2\alpha}$ that depends both on the goodness of the risky asset w.r.t. the riskless one and on the weight given to the minimization of the variance. Thus, the higher the Sharpe ratio of the risky asset, β , the higher the expected optimal final wealth, everything else being equal; the higher the member's risk aversion, α , the lower its mean. These are intuitive results.

Using (21), (10) and (8), it is possible to write $\bar{y}(t, x)$ in this way:

$$\bar{y}(t, x) = -\frac{\beta}{\sigma x} \left(x - \left(x_0 e^{rt} + \frac{c}{r}(e^{rt} - 1) \right) - \frac{e^{-r(T-t)+\beta^2 T}}{2\alpha} \right). \quad (22)$$

The amount $x\bar{y}(t, x)$ invested in the risky asset at time t is proportional to the difference between the fund x at time t and the fund that would be available at time t investing always only in the riskless asset, minus a term that depends on β^2, α and the time to retirement. The higher the weight α given to the minimization of the variance, the lower the amount invested in the risky asset, and

vice versa, which is an obvious result. Evidently, α is a measure of risk aversion of the individual: the higher α the higher her risk aversion. It is clear that a necessary and sufficient condition for the fund to be invested at any time t in the riskless asset is $\alpha = +\infty$: the (extreme) strategy of investing the whole portfolio in the riskless asset is optimal if and only if the risk aversion is infinite.

Using (21) and (22) one can express the optimal investment strategy in terms of the expected final wealth in the following way:

$$\bar{y}(t, x) = -\frac{\lambda - r}{\sigma^2 x} \left[x - \left(E[\bar{X}(T)]e^{-r(T-t)} - \frac{c}{r}(1 - e^{-r(T-t)}) \right) - \frac{e^{-r(T-t)}}{2\alpha} \right]. \quad (23)$$

The interpretation is that the amount $x\bar{y}(t, x)$ invested in the risky asset at time t is proportional to the difference between the fund x at time t and the amount that would be sufficient to guarantee the achievement of the expected value by adoption of the riskless strategy until retirement, minus a term that depends on α and the time to retirement.

In realistic situations, when the minimization of the variance plays a role in the investor's decisions, expressions (18) and (19) allow one to choose her own profile risk/reward. In fact, as in classical mean-variance analysis, it is possible to express the variance - or the standard deviation - of the final fund in terms of its mean. The subjective choice of the profile risk/reward becomes easier if one is given the efficient frontier of feasible portfolios.

It can be shown (see Appendix A) that the variance of the final wealth is

$$Var(\bar{X}(T)) = \frac{e^{-\beta^2 T}}{1 - e^{-\beta^2 T}} \left(\frac{e^{\beta^2 T} - 1}{2\alpha} \right)^2 = \frac{e^{\beta^2 T} - 1}{4\alpha^2}, \quad (24)$$

The variance is increasing if the Sharpe ratio increases, which is an expected result: in this case the investment in the risky asset is heavier, leading to higher variance. Obviously, the higher the risk aversion α , the lower the variance of the final fund, which is null if and only if $\alpha = +\infty$: in this case, the portfolio is entirely invested in the riskfree asset and

$$\bar{X}(T) = E(\bar{X}(T)) = \bar{x}_0.$$

The efficient frontier of portfolios is (see Appendix A):

$$E(\bar{X}(T)) = \bar{x}_0 + \left(\sqrt{e^{\beta^2 T} - 1} \right) \sigma(\bar{X}(T)). \quad (25)$$

Expectedly, the efficient frontier in the mean-standard deviation diagram is a straight line with slope $\sqrt{e^{\beta^2 T} - 1}$ which is called "price of risk" (see Luenberger (1998)): it indicates by how much the mean of the final fund increases if the volatility of the final fund increases by one unit. When $c = 0$, the efficient frontier coincides with that found by Richardson (1989), Bajeux-Besnainou and Portait (1998) and Zhou and Li (2000) for self-financing portfolios.

4 The expected utility approach: optimal portfolios for CARA and CRRA utility functions

4.1 The expected utility maximization problem

In this section, we assume that the individual solves her portfolio selection problem with the expected utility maximization approach. Therefore, her aim is now find the optimal investment strategy over

time that maximizes the expected value of final wealth. She then wants to solve

$$\begin{aligned} & \text{Maximize} \quad (J(y(\cdot))) \equiv E[U(X(T))], \\ & \text{subject to} \quad \begin{cases} y(\cdot) & \text{admissible} \\ X(\cdot), y(\cdot) & \text{satisfy (3).} \end{cases} \end{aligned} \quad (26)$$

Problem (26) is a standard optimization problem that can be dealt with via classical control theory. We refer the interest reader to classical texts such as Yong and Zhou (1999), Øksendal (1998), Björk (1998) and contain ourselves to a brief description of the basic steps to follow. One first defines a more general performance function

$$J(y(\cdot); t, x) = E_x[U(X(T))], \quad (27)$$

where $E_x[\cdot] = E[\cdot | X(t) = x]$, then defines the optimal value function as the supremum of the performance criterion among admissible controls,

$$V(t, x) := \sup_{y(\cdot)} J(y(\cdot); t, x). \quad (28)$$

Then, applying a fundamental theorem of stochastic control theory, writes the Hamilton-Jacobi-Bellman (HJB) equation that the value function associated to this problem must satisfy:

$$\sup_y \left[\frac{\partial V}{\partial t} + (x(y(\lambda - r) + r) + c) \frac{\partial V}{\partial x} + \frac{1}{2} x^2 \sigma^2 y^2 \frac{\partial^2 V}{\partial x^2} \right] = 0, \quad (29)$$

with boundary condition

$$V(T, x) = U(x). \quad (30)$$

Then, she writes the optimal control associated to the problem, as a function of partial derivatives of the value function:

$$y^*(t, x) = -\frac{\lambda - r}{\sigma^2 x} \frac{V_x}{V_{xx}}, \quad (31)$$

where $V_x = \frac{\partial V}{\partial x}$ and $V_{xx} = \frac{\partial^2 V}{\partial x^2}$, plugs (31) into the HJB equation to find the non-linear PDE

$$V_t + (rx + c)V_x - \frac{1}{2} \beta^2 \frac{V_x^2}{V_{xx}} = 0, \quad (32)$$

with boundary condition (30). By solving the PDE (30)-(32) one retrieves the optimal control via (31). The usual way to solve the non-linear PDE (32) is by guessing the form of the solution exploiting the natural similarity with the utility function selected. The guess technique works well with the utility functions considered in this paper, namely the exponential, the logarithmic and the power utility function. In each of the three cases, we report the expected value and the variance of terminal wealth. All derivations are in Appendix B.

4.2 CARA: Exponential utility function

Consider the exponential utility function

$$U(x) = -\frac{1}{k} e^{-kx}, \quad (33)$$

with (constant) Arrow-Pratt coefficient of absolute risk aversion equal to

$$ARA(x) = -\frac{U''(x)}{U'(x)} = k.$$

It can be shown (see Appendix B.1) that the expected final wealth is

$$E(X^*(T)) = \left(x_0 + \frac{c}{r}\right) e^{rT} - \frac{c}{r} + \frac{\beta^2 T}{k} = \bar{x}_0 + \frac{\beta^2 T}{k}, \quad (34)$$

and the variance of the final fund is

$$Var(X^*(T)) = E((X^*(T))^2) - E^2(X^*(T)) = \frac{\beta^2 T}{k^2}. \quad (35)$$

4.3 CRRA: Logarithmic utility function

Consider the logarithmic utility function

$$U(x) = \ln x. \quad (36)$$

The (constant) Arrow-Pratt coefficient of relative risk aversion is

$$RRA(x) = -\frac{U''(x)}{U'(x)} x = 1.$$

It can be shown (see Appendix B.2) that the expected final wealth is

$$E(X^*(T)) = e^{AT} \left(x_0 + \frac{c}{r}(1 - e^{-rT})\right) = \bar{x}_0 e^{\beta^2 T}, \quad (37)$$

and the variance of the final fund is

$$Var(X^*(T)) = (e^{KT} - e^{2AT}) \left(x_0 + \frac{c}{r}(1 - e^{-rT})\right)^2 = (E(X^*(T)))^2 (e^{\beta^2 T} - 1), \quad (38)$$

where

$$A = r + \beta^2, \quad K = 2r + 3\beta^2. \quad (39)$$

4.4 CRRA: Power utility function

Consider the power utility function

$$U(x) = \frac{x^\gamma}{\gamma},$$

with $\gamma < 1$ and (constant) Arrow-Pratt coefficient of relative risk aversion equal to

$$RRA(x) = -\frac{U''(x)}{U'(x)} x = 1 - \gamma.$$

It can be shown (see Appendix B.3) that the expected final wealth is

$$E(X^*(T)) = e^{AT} \left(x_0 + \frac{c}{r}(1 - e^{-rT})\right) = \bar{x}_0 e^{\frac{\beta^2 T}{1-\gamma}}, \quad (40)$$

and the variance of the final fund is

$$Var(X^*(T)) = (e^{KT} - e^{2AT}) \left(x_0 + \frac{c}{r}(1 - e^{-rT})\right)^2 = (e^{\frac{\beta^2 T}{(1-\gamma)^2}} - 1) (E(X^*(T)))^2, \quad (41)$$

where A and K are given by (39).

5 Mean-variance versus expected utility

The aim of this section is to prove that the optimal portfolios derived via maximization of expected utility of final wealth with the utility functions that exhibit constant absolute risk aversion and constant relative risk aversion are not efficient in terms of mean-variance. Before showing the procedure, it is convenient to recall some previous results. In section 3 we have shown that a member of a defined contribution pension scheme wanting to solve the following mean-variance problem

$$\min_{y(\cdot)} [-E(X(T)) + \alpha \text{Var}(X(T))] \quad (42)$$

where $\alpha > 0$ measures her risk aversion, should invest optimally in such a way as to obtain a final fund that has the following mean:

$$E(\bar{X}(T)) = \bar{x}_0 + \frac{e^{\beta^2 T} - 1}{2\alpha}, \quad (43)$$

and the following variance:

$$\text{Var}(\bar{X}(T)) = \frac{e^{\beta^2 T} - 1}{4\alpha^2}. \quad (44)$$

In other words, for this problem there exists no portfolio that has a final mean equal to (43) with a variance strictly lower than (44). Equivalently, there exists no portfolio that has a final variance equal to (44) with a mean strictly greater than (43).

In order to prove that the utility function U produces optimal portfolios that are not efficient, one can proceed along the following steps:

1. Derive the expectation and variance of final wealth under optimal control associated to the problem of maximization of $U(X(T))$, $E(X_U^*(T))$ and $\text{Var}(X_U^*(T))$.
2. Then, prove either

$$\begin{aligned} \text{(a)} \quad E(X_U^*(T)) = E(\bar{X}(T)) &\quad \Rightarrow \quad \text{Var}(X_U^*(T)) > \text{Var}(\bar{X}(T)); \\ \text{or} \\ \text{(b)} \quad \text{Var}(X_U^*(T)) = \text{Var}(\bar{X}(T)) &\quad \Rightarrow \quad E(X_U^*(T)) < E(\bar{X}(T)). \end{aligned}$$

As a byproduct, either the difference

$$\text{Var}(X_U^*(T)) - \text{Var}(\bar{X}(T)) > 0$$

or the difference

$$E(\bar{X}(T)) - E(X_U^*(T)) > 0$$

quantifies the degree of inefficiency of the utility function U . In the proof of Theorem 3 we will follow procedure (a).

We are now ready to state and prove the main result of this paper.

Theorem 3 *Assume that the financial market and the wealth equation are as described in section 2. Assume that the portfolio selection problem is solved via maximization of the expected utility of final wealth at time T , with preferences described by the utility function $U(x)$, as in section 4.1. Then, the couple $(\text{Var}(X_U^*(T)), E(X_U^*(T)))$ associated to the final wealth under optimal control $X_U^*(T)$ is*

not mean-variance efficient in the following cases:

i) $U(x) = -\frac{1}{k}e^{-kx}$;

ii) $U(x) = \ln x$;

iii) $U(x) = \frac{x^\gamma}{\gamma}$.

Proof. See Appendix C.

5.1 The special case $c = 0$: the usual portfolio selection problem

It is rather clear from the previous analysis, that the inequalities still hold in the three cases when $c = 0$. In this case, for the problem to be not trivial it must be $x_0 > 0$. Therefore we find that in typical portfolio selection analysis in continuous time, in a standard Black & Scholes financial market the expected utility maximization criterion with CARA and CRRA utility functions leads to an optimal portfolio that is not mean-variance efficient. We can summarize this result in the following corollary.

Corollary 4 *Assume that an investor wants to invest a wealth of $x_0 > 0$ for the time horizon $T > 0$ in a financial market as in section 2 and wealth equation (3) with $c = 0$. Assume that she maximizes expected utility of final wealth at time T . Then, the couple $(\text{Var}(X_U^*(T)), E(X_U^*(T)))$ associated to the final wealth under optimal control $X_U^*(T)$ is not mean-variance efficient in the following cases:*

i) $U(x) = -\frac{1}{k}e^{-kx}$;

ii) $U(x) = \ln x$;

iii) $U(x) = \frac{x^\gamma}{\gamma}$.

Proof. The proof is obvious, by setting $c = 0$ in the proof of Theorem (3) and observing that inequalities (113), (120) and (124), still hold. \square

6 Measure of inefficiency

The result proven in the previous section is not surprising. In fact, by definition the variance of a portfolio on the efficient frontier is lower or equal than the variance of any other portfolio with the same mean. However, what can be interesting is the extent of inefficiency of a portfolio found with the expected utility maximization approach, and its dependence on time horizon, risk aversion and financial market. This seems relevant for applicative purposes, considering the fact that EU approach with CARA and CRRA utility functions is widely used in the portfolio selection literature, also for long-term investment such as pension funds. One may well argue that if the individual's preferences are represented by, say, the power utility function, then she is not mean-variance optimizer, and she does not care not to be. This is fair. However, we observe three things. First, it is evidently difficult for an agent to specify her own utility function and the corresponding risk aversion parameter. On the contrary, it is relatively easy to reason in terms of targets to reach. This was observed also by Kahneman and Tversky (1979) in their classical paper on Prospect Theory. As will be shown later, the target-based approach is mean-variance efficient. Second, for most individuals it is rather immediate to understand the mean-variance criterion. It is indeed enough to show them two distributions of final wealth with same mean but different variances: in the context of pension funds, given that the final wealth refers to retirement saving, most workers would probably choose the distribution with lower variance. Third, the mean-variance criterion is still the most used criterion to

value and compare investment funds performances. We therefore believe that every further step in understanding the mean-variance approach should be encouraged in the context of pension schemes.

In this section, we define a measure of mean-variance inefficiency for a portfolio. As mentioned in section 5, the inefficiency of an optimal portfolio can be naturally measured by the difference between its variance and the corresponding minimal variance. We therefore define the Variance Inefficiency Measure as

$$VIM(X_U^*(T)) := Var(X_U^*(T)) - Var(\bar{X}(T)). \quad (45)$$

In each of the three cases considered, we analyze the dependence of the inefficiency measure on the relevant parameters of the problem, namely the risk aversion of the member, the Sharpe ratio β , the time horizon T , the initial wealth x_0 and the contribution rate c . We will perform the analysis of $VIM(X_U^*(T))$ for the three cases separately.

6.1 Exponential utility function

When

$$U(x) = -\frac{1}{k}e^{-kx},$$

we have

$$VIM(X^*(T)) = \frac{(e^{\beta^2 T} - 1)}{2\alpha k} \left(1 - \frac{k}{2\alpha}\right) = \frac{\beta^2 T}{k^2} \left(1 - \frac{\beta^2 T}{e^{\beta^2 T} - 1}\right). \quad (46)$$

So that

1. The inefficiency is a decreasing function of the absolute risk aversion coefficient $ARA = k$.
2. The inefficiency is an increasing function both of the Sharpe ratio β and the time horizon T .
3. The inefficiency does not depend on the initial fund x_0 and on the contribution rate c .

Let us make some comments on the extreme cases in which the two portfolios coincide and the inefficiency (46) is null. It is rather obvious that for $k \rightarrow +\infty$ the optimal portfolio is the riskless one, with mean \bar{x}_0 and zero variance, since the investor has infinite risk aversion. At the same time, due to (112), also $\alpha \rightarrow +\infty$ and the efficient portfolio is the riskless one.

Similarly, it is obvious that the difference in (46) is null also in the case $e^{\beta^2 T} = 1$. In fact, this is possible if either $\beta = 0$ or $T = 0$. In both cases, we have that the optimal portfolio is invested entirely in the riskless asset and the final deterministic portfolio at time $T \geq 0$ is \bar{x}_0 .

6.2 Logarithmic utility function

When

$$U(x) = \ln x,$$

we have

$$VIM(X^*(T)) = \bar{x}_0^2 (e^{\beta^2 T} - 1)^2 (e^{\beta^2 T} + 1). \quad (47)$$

So that

1. The inefficiency is an increasing function both of the Sharpe ratio β and the time horizon T .

2. The inefficiency is an increasing function of both the initial fund $x_0 \geq 0$ and the contribution rate $c \geq 0$.

Given that $\bar{x}_0 > 0$ for the problem not to be trivial, the difference in (47) is null if and only if $e^{\beta^2 T} = 1$. As observed earlier, this is possible if either $\beta = 0$ or $T = 0$. In both cases, we have that the optimal portfolio is invested entirely in the riskless asset and the final portfolio at time $T \geq 0$ is \bar{x}_0 .

6.3 Power utility function

When

$$U(x) = \frac{x^\gamma}{\gamma},$$

we have

$$VIM(X^*(T)) = \frac{\bar{x}_0^2}{e^{\beta^2 T} - 1} \left(e^{\frac{2\beta^2 T}{1-\gamma}} (e^{\frac{\beta^2 T}{(1-\gamma)^2}} - 1)(e^{\beta^2 T} - 1) - (e^{\frac{\beta^2 T}{1-\gamma}} - 1)^2 \right). \quad (48)$$

With the change of variables:

$$b := e^{\beta^2 T} \quad a := \frac{1}{1-\gamma},$$

we have:

$$VIM(X^*(T)) = VIM(a, b) = \frac{\bar{x}_0^2}{b-1} (b^{a^2+2a+1} - b^{a^2+2a} - b^{2a+1} + 2b^a - 1). \quad (49)$$

From the proof of Theorem (3), point iii), it is rather clear that

$$\frac{\partial VIM}{\partial a} > 0. \quad (50)$$

What is more difficult to prove is that

$$\frac{\partial VIM}{\partial b} > 0 \quad (51)$$

for all values of $a > 0$, so that is still an open problem. We are able to prove it for $b > 2 \wedge a > a_*$ where $a_* \simeq 0.45$ is the positive root of $2a^3 + 4a^2 - 1 = 0$. These values imply $RRA < \frac{1}{a_*} \simeq 2.22$ and $\beta^2 T > \ln 2 \simeq 0.69$. Our conjecture is that (51) holds for all possible values of $a > 0$, $b > 1$, but this is still to be proved. In conclusion, we have

1. The inefficiency is a decreasing function of the relative risk aversion coefficient $RRA = 1 - \gamma$.
2. The inefficiency is an increasing function both of the Sharpe ratio β and the time horizon T , if $\beta^2 T > \ln 2$ and $a < a_*$, where $a_* > 0$ solves $2a^3 + 4a^2 - 1 = 0$.
3. The inefficiency is an increasing function of both the initial fund $x_0 > 0$ and the contribution rate $c > 0$.

One can see that for $\gamma \rightarrow -\infty$ the optimal portfolio is the riskless one, with mean \bar{x}_0 and zero variance, since the investor has infinite risk aversion. At the same time, due to (123), also the efficient portfolio will be the riskless one. Therefore in this case, the difference in (48) is null.

Similarly, one can see that the difference in (48) is null also in the case $e^{\beta^2 T} = 1$. In fact, this is possible if either $\beta = 0$ or $T = 0$. In both cases, we have that the optimal portfolio is invested entirely in the riskless asset and the final deterministic portfolio at time $T \geq 0$ is \bar{x}_0 .

7 Quadratic loss function: the target-based approach

In this section, we show the expected result that in the framework outlined in section 2 the quadratic utility-loss function is consistent with the mean-variance approach. Most of the results of this section can be found in Højgaard and Vigna (2007). Since in this paper the focus is on portfolio choices in defined contribution pension schemes, we will consider a modified version of the simple quadratic utility function, considering a target-based approach induced by a quadratic loss function. Optimization of quadratic loss or utility function is a typical approach in pension schemes. Examples of this approach can be found for instance in Boulier, Michel and Wisnia (1996), Boulier, Trussant and Florens (1995), Cairns (2000), Haberman and Sung (1994) for defined benefit pension funds, in Haberman and Vigna (2002), Gerrard, Haberman and Vigna (2004), Gerrard, Højgaard and Vigna (2010) for defined contribution pension schemes.

Højgaard and Vigna (2007) consider the problem of a member of a DC pension scheme who chooses a target value at retirement F and chooses the optimal investment strategy that minimizes¹

$$E [(X(T) - F)^2].$$

In these circumstances, we shall say that the member solves the portfolio selection problem with the target-based (T-B) approach. For the problem to be financially interesting, the final target F should be chosen big enough, i.e. such that

$$F > \bar{x}_0. \quad (52)$$

We can see from Gerrard et al. (2004) that the optimal investment strategy for the T-B approach is given on the following form²

$$y_{tb}(t, x) = -\frac{\lambda - r}{\sigma^2 x}(x - G(t)), \quad (53)$$

where

$$G(t) = Fe^{-r(T-t)} - \frac{c}{r}(1 - e^{-r(T-t)}), \quad (54)$$

and

$$y_{tb}(t, x) = -\frac{\lambda - r}{\sigma^2 x} \left[x - \left(Fe^{-r(T-t)} - \frac{c}{r}(1 - e^{-r(T-t)}) \right) \right]. \quad (55)$$

Let us notice that the function $G(t)$ represents a sort of target level for the fund at time t : should the fund $X(t)$ reach $G(t)$ at some point of time $t < T$, then the final target F could be achieved by adoption of the riskless strategy until retirement. However, as will be shown, the achievement of $G(t)$ and therefore the sure achievement of the target, is prevented under optimal control by the construction of the solution.

In order to prove efficiency of the T-B approach, we now need to set the expected value of final wealth under optimal control equal to that of the mean variance approach. To calculate the

¹In a more general model, presented in Gerrard et al. (2004), the individual chooses a target function $F(t)$ so as to minimize

$$E \left[\int_0^T e^{-\rho t} \varepsilon_1 (X(t) - F(t))^2 dt + \varepsilon_2 e^{-\rho T} (X(T) - F(T))^2 \right]$$

Here, for consistent comparisons we eliminate the running cost and select only the terminal wealth problem.

²Notice that Gerrard et al. (2004) consider the decumulation phase of a DC scheme. The difference in the wealth equation is that in that case there are periodic withdrawals from the fund whereas here we have periodic inflows into the fund. Formally the equations are identical if one sets $-b_0 = c$.

expected value of the final fund in the T-B approach we let $X^*(t)$ denote the optimal wealth function for this case. Then in Gerrard et al. (2004) it can be seen that $X^*(t)$ satisfies the following SDE:

$$dX^*(t) = [rG(t) + c + (\beta^2 - r)(G(t) - X^*(t))]dt + \beta(G(t) - X^*(t))dW(t). \quad (56)$$

As in previous work, let us define the process

$$Z(t) = G(t) - X^*(t). \quad (57)$$

Then

$$dZ(t) = G'(t)dt - dX^*(t) = (r - \beta^2)Z(t)dt - \beta Z(t)dW(t), \quad (58)$$

where in the last equality we have applied (54) and (56). We can see that the process $Z(t)$ follows a geometric Brownian motion and is given by:

$$Z(t) = Z(0)e^{(r - \frac{3}{2}\beta^2)t - \beta W(t)}. \quad (59)$$

Noting that

$$G(T) = F,$$

one has

$$Z(T) = F - X^*(T).$$

Thus

$$E(X^*(T)) = F - E(Z(T)) = F - (G(0) - x_0)e^{-(\beta^2 - r)T} = e^{-\beta^2 T}\bar{x}_0 + (1 - e^{-\beta^2 T})F. \quad (60)$$

The expected final fund turns out to be a weighted average of the target and of the fund that one would have by investing fully in the riskless asset. Furthermore, it is straightforward to see that in the T-B approach the final target cannot be reached. In fact, from (59), one can see that $Z(t) > 0$ for $t \leq T$ if $Z(0) > 0$. Let us notice that this holds, due to (52). In fact,

$$Z(0) = G(0) - x_0 = Fe^{-rT} - \frac{c}{r}(1 - e^{-rT}) - x_0 = e^{-rT}(F - \bar{x}_0) > 0. \quad (61)$$

Therefore, the final fund is always lower than the target. This result is not new. A similar result was already found by Gerrard et al. (2004) and by Gerrard, Haberman and Vigna (2006) in the decumulation phase of a DC scheme: with a different formulation of the optimization problem and including a running cost, in both works they find that there is a "natural" time-varying target that acts as a sort of safety level for the needs of the pensioner and that cannot be reached under optimal control. Previously, in a different context, a similar result was found by Browne (1997): in a problem where the aim is to maximize the probability of hitting a certain upper boundary before ruin, when optimal control is applied the safety level (the minimum level of fund that guarantees fixed consumption by investing the whole portfolio in the riskless asset) can never be reached.

We are now ready to state and prove a theorem that shows that the target-based approach is mean-variance efficient and that each point on the efficient frontier corresponds to the optimal solution of a T-B optimization problem.

Theorem 5 *Assume that the financial market and the wealth equation are as described in section 2. Assume that the portfolio selection problem is solved via minimization of expected loss of final wealth at time T , with preferences described by the loss function $L(x)$. Then,*

i) the couple $(\text{Var}(X_L^(T)), E(X_L^*(T)))$ associated to the final wealth under optimal control $X_L^*(T)$ is mean-variance efficient if $L(x) = (F - x)^2$;*

ii) each point $(\text{Var}(\bar{X}(T)), E(\bar{X}(T)))$ on the efficient frontier as outlined in section 3 is the solution of an expected loss minimization problem with loss function $L(x) = (F - x)^2$.

Proof. See Appendix D.

Thus, every solution to a target-based optimization problem corresponds to a point on the efficient frontier, and each point of the efficient frontier can be found by solving a target-based optimization problem. The one-to-one correspondence between points of the efficient frontier and target-based optimization problems is given by the following relationship between the parameter α of the mean-variance approach and the value of final target of the target-based approach:

$$\alpha = \frac{e^{\beta^2 T}}{2(F - \bar{x}_0)}, \quad (62)$$

where we have used (60) and (157).

The fact that the target-based approach is a particular case of the mean-variance approach should put an end to the criticism of the quadratic utility function, that penalizes deviations above the target as well as deviations below it. The intuitive motivation for supporting such a utility function in DC schemes: "The choice of trying to achieve a target and no more than this has the effect of a natural limitation on the overall level of risk for the portfolio: once the target is reached, there is no reason for further exposure to risk and therefore any surplus becomes undesirable" finds here full justification in a rigorous setting.

We notice that a similar result was mentioned, without proof, by Bielecky et al. (2005). They noticed, however, that the portfolio's expected return would be unclear to determine a priori. In contrast, here we provide the exact expected return and variance of the optimal portfolio via optimization of the quadratic loss function. We are thus able to determine completely the exact point on the efficient frontier of portfolios.

A final remark about an intrinsic feature of the optimal efficient investment strategies. From (53) we can see that another direct consequence of the positivity of $Z(t)$ is the fact that under the target-based approach the amount invested in the risky asset under optimal control is always positive. Obviously, this is the case also for the mean-variance approach. This leads us to the formulation of the following corollary.

Corollary 6 *Consider the financial market and the wealth equation as in section 2. Consider the efficient frontier of feasible portfolios, as outlined in section 3. Then, the optimal amount invested in the risky asset at any time $0 \leq t < T$ is strictly positive.*

Proof. This follows from (159), (57), (59) and (61). \square

This is a desirable property, given that the constrained portfolio problem has not been solved yet for the target-based approach. In fact, this natural feature allows to reduce the bilateral constrained portfolio problem in the no-borrowing constraint problem, given that the no-short selling comes with no cost for the nature of the problem. Solving the no-short selling constrained problem with the target-based approach in the decumulation phase of a defined contribution pension scheme is subject of ongoing research (see Di Giacinto, Federico, Gozzi and Vigna (2009a)).

8 Numerical application

8.1 The efficient frontier

In this section, with a numerical example we intend to illustrate the extent of inefficiency of optimal portfolios for DC pension schemes whenever CARA and CRRA utility functions are used to solve

the portfolio selection problem. We will do this by comparing optimal inefficient portfolios with the corresponding mean-variance efficient one. For illustrative purposes, we will also report results for the lifestyle strategy (see e.g. Cairns et al. (2006)), widely used by DC pension plans in UK. In the lifestyle strategy the fund is invested fully in the risky asset until 10 years prior to retirement, and then is gradually switched into the riskless asset by switching 10% of the portfolio from risky to riskless asset each year. The parameters for asset returns are as in Højgaard and Vigna (2007), i.e. $r = 0.03$, $\lambda = 0.08$, $\sigma = 0.15$, $c = 0.1$, $x_0 = 1$, $T = 20$. Therefore, the Sharpe ratio is $\beta = 0.33$ and the fund achievable under the riskless strategy is $\bar{x}_0 = 4.56$. The comparison will be done for each of the three inefficient utility functions, considering appropriate values for the risk aversion displayed.

It is far beyond the scope of this paper to discuss the choice of appropriate values for the parameters of absolute and relative risk aversion for the exponential and the power utility function. However, we notice that while there seems to be overall agreement across the literature regarding typical values of the RRA coefficient, this is not the case for the choice of the ARA coefficient. In addition, there seems to be little evidence of constant absolute risk aversion displayed by investors (see for instance, Guiso and Paiella (2008)). The value of $ARA = 20$ used by Battocchio and Menoncin (2004) is not appropriate in this context, because it would imply an α value of around 37, with implied final target $F = 4.67$, too much close to the basic value achievable with the riskless strategy, $\bar{x}_0 = 4.56$. Therefore, such high values of k , used also elsewhere in the literature (see for instance Jorion (1985)) have to be considered too high in this model with this time horizon. On the other hand, Guiso and Paiella (2008) suggest that the average absolute risk aversion should range around 0.02, a too low value for this context, implying a final target of $F = 129$, clearly unreasonable. We have then decided to test different levels of risk aversion for the power case, as in many previous works of this kind. We will be considering $RRA=1$ (logarithmic utility), $RRA=2$ and $RRA=5$. However, in each case we will report the corresponding results also for the exponential utility function, as implied by the choice of the relative risk aversion. The choice of $RRA = 2$ is motivated by the evident consensus in the literature regarding constant relative risk aversion coefficient of about 2. See, for instance Schlechter (2007), who sets a minimum bound of around 1.92 with no savings, and of 2.42 in the presence of savings. More specifically, regarding active members of pension schemes Canessa and Dorich (2008) in a recent survey reported an overall average of relative risk aversion of about 1.81, depending on the age of the group under investigation. In particular, the RRA coefficient of the group under study varies between 1.59 and 1.88 for younger members, and between 2.21 and 2.25 for older ones. The choice of $RRA = 5$, motivated by the importance of showing results relative to higher risk aversion, is in line with similar choices for DC pension plans members (see Cairns et al. (2006)) and is consistent with the choice of the final target operated by Højgaard and Vigna (2007). Not least, $RRA=5$ gives an expected final fund very similar to that empirical found by application of the lifestyle strategy (see later) and therefore allows consistent comparisons.

We have then the following three cases:

- low risk aversion: $RRA = 1$, that is the logarithmic utility function, implies $\alpha = 0.1096$, which in turn leads to $F = 46.66$; this corresponds to $k = 0.059$ in the exponential model;
- medium risk aversion: $RRA = 2$ implies $\alpha = 0.44$, which in turn leads to $F = 14.99$; this corresponds to $k = 0.24$ in the exponential model;
- high risk aversion: $RRA = 5$ implies $\alpha = 1.61$, which in turn leads to $F = 7.43$; this corresponds to $k = 0.87$ in the exponential model.

Table 1 reports for each value of the RRA coefficient the corresponding α value, the corresponding target in the T-B approach, the corresponding coefficient of absolute risk aversion k , mean and

variance of the efficient portfolio, variance of the optimal portfolio for power and exponential utility function and VIM (Variance Inefficiency Measure) for both utility functions. Clearly, when $RRA=1$ the power degenerates in the logarithmic utility function.

Remark 1

Notice that in the fifth column the MV efficient expected value $E(\bar{X}(T))$ coincides with the expected value associated to the power and exponential optimal portfolios, $E(X^*(T))$. In the label we report only $E(\bar{X}(T))$ for space constraints.

RRA $1 - \gamma$	α	Target F	ARA k	MV efficient $E(\bar{X}(T))$	MV efficient $\text{Var}(\bar{X}(T))$	Power $\text{Var}(X^*(T))$	Exponential $\text{Var}(X^*(T))$	Power VIM	Exponential VIM
1	0.11	46.66	0.06	42.1	171	14584	634	14413	463
2	0.44	14.99	0.24	13.86	11	143	39	132	28
5	1.61	7.43	0.87	7.12	0.79	4.71	2.93	3.91	2.14

Table 1.

It is evident the extent of inefficiency when the risk aversion is too low. Namely, the VIM in the logarithmic case is 14413 and when $ARA = 0.06$ VIM in the exponential case is 463. More in general, one can observe that the inefficiency decreases when RRA and ARA increase. This comes directly from results shown in section 6. We also observe the interesting feature that in every choice for the relative risk aversion coefficient, the inefficiency produced by the exponential utility function is lower than that of the power utility.

Figures 1, 2 and 3 report in the usual standard deviation/mean diagram the efficient frontier and the optimal portfolios in the cases $RRA=1, 2, 5$, respectively. In Figure 3, we have added also the empirical portfolio obtained via adoption of the lifestyle strategy. In order to find it, we have carried out 1000 Monte Carlo simulations with discretization done on a weekly basis and in each scenario have applied the lifestyle strategy described before and obtained the final wealth. We have then plotted the point with coordinates equal to standard deviation and mean of the distribution of final wealth over the 1000 scenarios. Noticing that the mean of the final wealth for the lifestyle is 7.31, we have plotted it only in Figure 3, that reports results for $RRA=5$, with mean equal to 7.12. For completeness of exposition, Table 2 reports for each risk aversion coefficient, the standard deviation of each optimal portfolio, that is the x -coordinate of the optimal point in the Figures, the y -coordinate being the mean, reported in the last column. As before, here Remark 1 applies.

RRA $1 - \gamma$	MV $\sigma(\bar{X}(T))$	Power $\sigma(X^*(T))$	Exponential $\sigma(X^*(T))$	MV efficient $E(\bar{X}(T))$
1	13.08	120.76	25.18	42.1
2	3.32	11.96	6.24	13.86
5	0.89	2.17	1.71	7.12

Table 2.

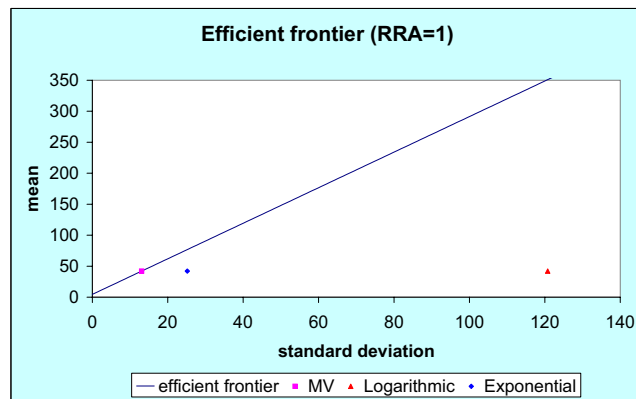


Figure 1.

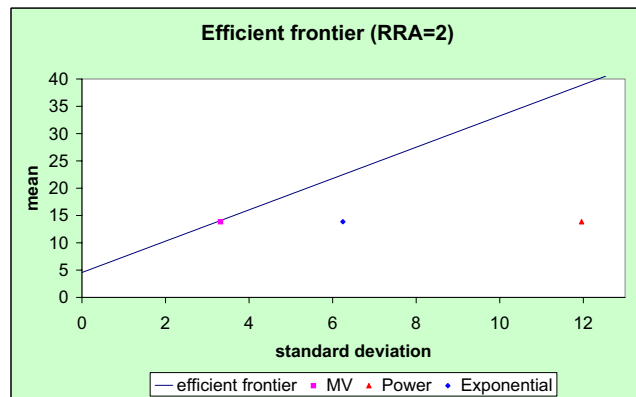


Figure 2.

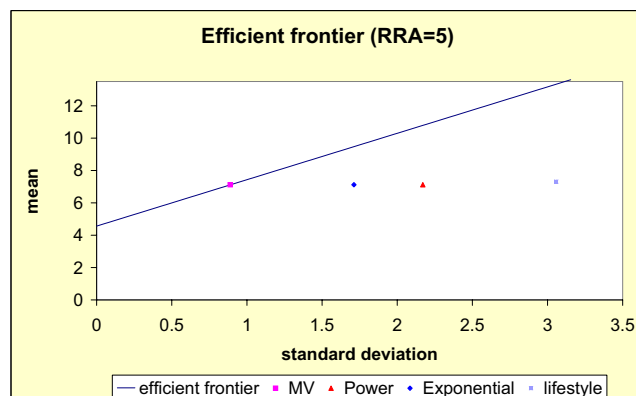


Figure 3.

The inefficiency for the logarithmic utility function is striking. This can be explained by observing that the inefficiency for the logarithmic utility function (47) is cubic in $e^{\beta^2 T}$, whereas it is linear in $e^{\beta^2 T}$ for the exponential case. Thus, with a high value of $\beta^2 T$ the inefficiency of the logarithmic function becomes more evident. This suggests that the logarithmic utility function is not appropriate for long time horizons or for high Sharpe ratios. As noted already, the exponential utility function is less inefficient than the power utility function and we observe that the lifestyle strategy is the most inefficient strategy. That the lifestyle strategy is mean-variance inefficient was already found by Højgaard and Vigna (2007). However, here we provide a more complete picture by measuring the inefficiency also for the most popular utility functions used for the portfolio selection of a DC scheme.

8.2 Numerical simulations with suboptimal policies

Figures 1, 2 and 3 or the values of the VIM certainly shed light on the extent of inefficiency and comparison between different portfolios. It is also possible to provide useful insight about the practical consequences of inefficiency by deriving in a simulations framework the distribution of the final fund. Thus, it is a useful exercise to carry out simulations for the risky asset, and see how the inefficiency translates into distribution of final wealth. We now focus only on $RRA = 5$ and carry out 1000 Monte Carlo simulations for the mean-variance approach, the power and the exponential utility functions. For consistent comparisons, for each of the four investment strategies tested (including results also for the lifestyle strategy) we have created the same 1000 scenarios, by applying in each case the same stream of pseudo random numbers.

As in Højgaard and Vigna (2007), we see that all optimal investment strategies tend to apply a remarkable amount of borrowing for small values of x . Since borrowing is likely to be ruled out by the scheme itself or by the legislation, we introduce applicable suboptimal strategies which are cut off at 0 or 1 if the optimal strategy goes beyond the interval $[0, 1]$. For this reason, in the tables and figures that follow we will name each strategy adding the word "cut". It must be mentioned that suboptimal policies of the same type were applied by Gerrard et al. (2006) in the decumulation phase of a DC scheme, and proved to be satisfactory: with respect to the unrestricted case, the effect on the final results turned out to be negligible and the controls resulted to be more stable over time. Clearly, imposing restriction on the controls would change substantially the formulation of the problem and would make it very difficult to tackle mathematically. Up to our knowledge, the only work where an optimization problem with constraints has been thoroughly treated in the accumulation phase of a DC scheme is Di Giacinto et al. (2009b).

Table 3 reports for the four strategies considered some percentiles of the distribution of the final wealth, its mean and standard deviation, the probability of reaching the target and the mean shortfall, defined as the mean of the deviation of the fund from the target, given that the target is not reached. We recall that the target in this case is 7.43. Figure 4 plots the suboptimal portfolios for the four strategies considered together with the efficient frontier.

Final wealth	MV cut	Power cut	Exponential cut	Lifestyle
5th perc.	3.65	4.05	4.05	3.8
25th perc.	6.36	5.28	5.6	5.13
50th perc.	7.1	6.45	6.71	6.61
75th perc.	7.32	7.93	7.88	8.72
95th perc.	7.4	10.63	9.57	13.57
Mean	6.54	6.78	6.77	7.32
Standard deviation	1.22	2.05	1.68	3.06
Prob reaching target	0	0.31	0.34	0.45
Mean shortfall	0.88	1.76	1.61	1.7

Table 3. Target $=F = 7.43$.

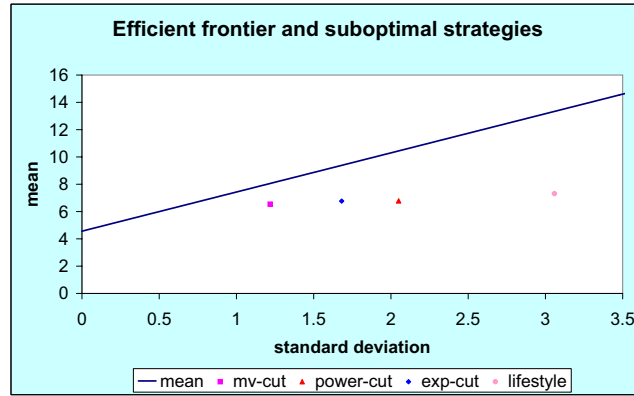


Figure 4.

A few comments can be gathered from Table 3 and Figure 4.

- Maybe the most important result is evident from Figure 4: the strategy which is most close to the efficient frontier is the mv-cut, followed by the exponential-cut, followed by the power-cut and then by the largely inefficient lifestyle strategy.
- The lifestyle strategy proves to be very far from being efficient. In particular, for being efficient it should provide either a standard deviation of about 0.96 (instead of 3.06) with same level of mean, or a mean of 13.34 (instead of 7.32) with the same level of standard deviation.
- The mv-cut, power-cut and exponential-cut provide, as expected, almost the same mean, but the mv-cut has a standard deviation much lower than that of the other two strategies. This can be found also by inspection of the percentiles of final wealth: in the mv-cut strategy in 75% of the scenarios the final wealth lies between 6.36 and 7.423 (that is the maximum value, not reported in Table 3). Considering that the target is 7.426, we find this is a satisfactory result.
- The fact that in the mv-cut strategy the target is never reached in these simulations is due to the target-based approach formulation: we have already shown in section 7 that with optimal policies the target can be approached very closely but can never be reached. Closeness to

the target would improve with a higher Sharpe ratio of the risky asset: Højgaard and Vigna (2007) find that with a Sharpe ratio of 0.5 in 75% of the scenarios the final wealth lies between 6.67 and 6.79, with a target of 6.78.

- The much lower dispersion of the mv-cut has as a direct consequence also on the mean shortfall value: the target is never reached, but the average distance from it is rather small, namely 0.88 which is 12% of the target. This is not the case for the power-cut and the exponential-cut strategies: in the former (latter) case the target is not reached in 69% (66%) of the cases with a mean shortfall of 1.76 (1.61), that amounts to 24% (22%) of the target.

As a final comment, we would like to add that it is certainly true that the higher dispersion of the exponential-cut and power-cut with respect to the mv-cut strategy means also a longer right tail of the distribution of final wealth, implying possibility of exceeding the target in about 30% of the cases. However, we believe that most active members of a pension scheme would not be willing to accept a significantly higher reduction in targeted wealth in exchange of having the chance of exceeding the targeted wealth in 30% of the cases. Therefore, we believe that the mean-variance (or target-based) approach should be preferred to expected utility maximization for the portfolio selection in defined contribution pension schemes, whenever CRRA or CARA utility functions were to be used. This conclusion is likely to apply also with other choices of market and risk aversion parameters, though here we have not tested robustness in order to limit the length of the paper.

9 Conclusions and further research

In this paper, we have compared expected utility and mean-variance approaches for portfolio selection in the accumulation phase of a defined contribution pension scheme. First, we have derived the optimal and efficient investment strategy with the mean-variance approach in continuous time, following the approach pioneered by Zhou and Li (2000). Then, we have found the optimal investment strategy with the expected utility maximization criterion, selecting CARA and CRRA utility functions. As the main result, we have shown that the optimal portfolios derived with these utility functions are not efficient in the mean-variance setting. Namely, the variance of the final wealth under optimal control is not the minimal variance. As a corollary, we have shown that these results hold also when the contribution rate is null, i.e. for the typical portfolio selection problem in continuous time in the Black-Scholes model.

We have then proposed a natural measure of inefficiency of optimal portfolios based on the difference between optimal portfolio variance and minimal variance. We have established dependence of such inefficiency measure from relevant parameters, such as risk aversion, Sharpe ratio of the risky asset, time horizon, initial wealth and contribution rate. We have proven the reasonable result that the inefficiency is a decreasing function of risk aversion. With CARA and logarithmic utility functions it is an increasing function of the Sharpe ratio and the time horizon, and the same applies for the power utility function, with not too high risk aversion and not too low time horizon and Sharpe ratio. We conjecture that the same result holds with the power utility function for all values of the parameters. With CARA utility function the efficiency does not depend on initial wealth and contribution rate, whereas with CRRA utility functions it is an increasing function of both. Finally, we have shown the expected result that the optimal portfolio derived by minimization of a quadratic target-based loss function (target-based approach) is mean-variance efficient. We have closed with a numerical application aimed at showing the extent of inefficiency for an active member of a defined contribution pension scheme.

Considering that investment in a pension fund is for retirement savings and is for long-term horizon, we doubt that most active members of pension schemes would be willing to accept a significantly higher reduction in targeted final wealth as the price to pay for having a chance to exceed the targeted wealth in some cases. Therefore, our conclusion is that the mean-variance (or target-based) approach should be preferred to expected utility maximization for the portfolio selection in defined contribution pension schemes, whenever CRRA or CARA utility functions are to be used. To further support our view, we observe three things. First, it is evidently difficult for an agent to specify her own utility function and the corresponding risk aversion parameter. On the contrary, it is relatively easy to reason in terms of targets to reach. This was observed also by Kahneman and Tversky (1979) in their classical paper on Prospect Theory and more recently by Bordley and Li Calzi (2000). Second, for most individuals it is rather immediate to understand the mean-variance criterion. It is indeed enough to show them two distributions of final wealth with same mean and different variances: in the context of pension funds, most workers would probably choose the distribution with lower variance. Third, the mean-variance criterion is still the most used criterion to value and compare investment funds performances: it is evidently appreciable if member and investment manager pursue the same goal.

This paper contributes also to the portfolio selection literature. To the best of our knowledge, a complete and rigorous comparison between the two leading approaches in continuous time portfolio selection, the mean-variance approach and expected utility maximization, has not been performed in the existing literature, although related work can be found in Zhou (2003). Now that the connection between the mean-variance approach and standard LQ control problems has been rigorously established, the quite rich stochastic control arsenal can be exploited to investigate further the comparison between the two leading methodologies for portfolio selection. This paper can be considered as a first step in this direction.

This work leaves ample scope for further research. Clearly, we need to consider a model with time-dependent drift and volatility. The extension to the multi-period discrete time framework is also appealing. Finally, the inclusion of a stochastic interest rate in the financial market is also worth exploring. Namely, a financial market that includes bond assets is crucial in a long time horizon context such as pension funds. In addition, this extension would be in line with the most advanced models for portfolio allocation in pension funds (see, for instance, Battocchio and Menoncin (2004), Boulier et al. (2001), Cairns et al. (2006), Deelstra et al. (2003), Gao (2008)). Therefore, this challenging task is in the agenda for future research.

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Appendix

A The efficient frontier

To find the efficient frontier of portfolios let us introduce the following notation:

$$\begin{aligned} y_0 &\equiv x_0 + \frac{c}{r} & \theta &\equiv 1 - e^{-\beta^2 T} \\ \rho &\equiv e^{-(\beta^2 - r)T} & \phi &\equiv e^{-(\beta^2 - 2r)T}. \end{aligned} \quad (63)$$

From (18) and (19), we have:

$$E(\bar{X}(T)) = y_0 \rho - \frac{c}{r}(1 - \theta) + \delta \theta, \quad (64)$$

and

$$E(\bar{X}^2(T)) = y_0^2 \phi - 2\frac{c}{r}y_0 \rho + \frac{c^2}{r^2}(1 - \theta) + \delta^2 \theta. \quad (65)$$

Therefore

$$\begin{aligned} Var(\bar{X}(T)) &= E(\bar{X}^2(T)) - E(\bar{X}(T))^2 = y_0^2 \phi - 2\frac{c}{r}y_0 \rho + \frac{c^2}{r^2}(1 - \theta) + \delta^2 \theta - \\ &- y_0^2 \rho^2 - \frac{c^2}{r^2}(1 - \theta)^2 - \delta^2 \theta^2 + 2y_0 \rho \frac{c}{r}(1 - \theta) - 2y_0 \rho \delta \theta + 2\frac{c}{r}(1 - \theta)\delta \theta. \end{aligned} \quad (66)$$

After a few passages and noticing that

$$\phi - \rho^2 = \phi \theta, \quad (67)$$

we have

$$Var(\bar{X}(T)) = y_0^2 \theta \phi + \theta(1 - \theta) \left(\delta + \frac{c}{r} \right)^2 - 2y_0 \rho \theta \left(\delta + \frac{c}{r} \right). \quad (68)$$

From (64), we have

$$\theta \left(\delta + \frac{c}{r} \right) = E(\bar{X}(T)) - y_0 \rho + \frac{c}{r}. \quad (69)$$

Therefore

$$\begin{aligned} Var(\bar{X}(T)) &= y_0^2 \theta \phi + \theta(1 - \theta) \frac{(E(\bar{X}(T)) - y_0 \rho + \frac{c}{r})^2}{\theta^2} - 2y_0 \rho \left(E(\bar{X}(T)) - y_0 \rho + \frac{c}{r} \right) \\ &= \frac{1 - \theta}{\theta} \left[\frac{y_0^2 \phi \theta^2}{1 - \theta} + \left(E(\bar{X}(T)) - y_0 \rho + \frac{c}{r} \right)^2 - \frac{2y_0 \rho \theta}{1 - \theta} \left(E(\bar{X}(T)) - y_0 \rho + \frac{c}{r} \right) \right] \\ &= \frac{1 - \theta}{\theta} \left[\frac{\phi \theta^2 + \rho^2 + \rho^2 \theta}{1 - \theta} y_0^2 + 2E(\bar{X}(T)) \frac{c}{r} + \frac{c^2}{r^2} + E(\bar{X}(T))^2 - 2y_0 \frac{\rho}{1 - \theta} \left(E(\bar{X}(T)) + \frac{c}{r} \right) \right]. \end{aligned} \quad (70)$$

Now notice that

$$\frac{\phi \theta^2 + \rho^2 + \rho^2 \theta}{1 - \theta} = e^{2rT} \quad \text{and} \quad \frac{\rho}{1 - \theta} = e^{rT}. \quad (71)$$

So that

$$\begin{aligned} Var(\bar{X}(T)) &= \frac{1 - \theta}{\theta} \left[y_0^2 e^{2rT} + \left(E(\bar{X}(T)) + \frac{c}{r} \right)^2 - 2y_0 e^{rT} \left(E(\bar{X}(T)) + \frac{c}{r} \right) \right] \\ &= \frac{1 - \theta}{\theta} \left[\left(E(\bar{X}(T)) + \frac{c}{r} \right) - y_0 e^{rT} \right]^2 \\ &= \frac{e^{-\beta^2 T}}{1 - e^{-\beta^2 T}} \left[E(\bar{X}(T)) - \left(x_0 e^{rT} + c \frac{e^{rT} - 1}{r} \right) \right]^2 \\ &= \frac{e^{-\beta^2 T}}{1 - e^{-\beta^2 T}} \left[E(\bar{X}(T)) - \bar{x}_0 \right]^2 \\ &= \frac{e^{-\beta^2 T}}{1 - e^{-\beta^2 T}} \left(\frac{e^{\beta^2 T} - 1}{2\alpha} \right)^2 = \frac{e^{\beta^2 T} - 1}{4\alpha^2}, \end{aligned} \quad (72)$$

where in the forth last equality we have used (63).

B Derivation of expected values and variances with EU approach

B.1 Exponential utility function

The value function of this problem is (see for instance Devolder et al. (2003)):

$$V(t, x) = -\frac{1}{k} \exp[-k(a(t) + b(t)(x - f(t)))], \quad (73)$$

with

$$\begin{aligned} a(t) &= \frac{\beta^2}{2k}(T - t), \\ b(t) &= e^{r(T-t)}, \\ f(t) &= \frac{c}{r}(e^{-r(T-t)} - 1). \end{aligned}$$

The optimal amount invested in the risky asset at time t if the wealth is x is:

$$xy^*(t, x) = \frac{\beta}{\sigma k} e^{-r(T-t)}. \quad (74)$$

The evolution of the fund under optimal control $X^*(t)$ is given by:

$$dX^*(t) = \left[\frac{\beta^2}{k} e^{-r(T-t)} + xr + c \right] dt + \frac{\beta}{k} e^{-r(T-t)} dW(t). \quad (75)$$

By application of Ito's lemma to (75) the evolution of its square, $(X^*(t))^2$ is given by

$$d(X^*(t))^2 = \left[2r(X^*(t))^2 + \left(2c + \frac{2\beta^2}{k} e^{-r(T-t)} \right) X^*(t) + \frac{\beta^2}{k^2} e^{-2r(T-t)} \right] dt + \frac{2\beta X^*(t)}{k} e^{-r(T-t)} dW(t). \quad (76)$$

If we take expectations on lhs and rhs in both (75) and (76) we have the following ODEs:

$$dE(X^*(t)) = \left[rE(X^*(t)) + \left(c + \frac{\beta^2}{k} e^{-r(T-t)} \right) \right] dt, \quad (77)$$

$$dE((X^*(t))^2) = \left[2rE((X^*(t))^2) + \left(2c + \frac{2\beta^2}{k} e^{-r(T-t)} \right) E(X^*(t)) + \frac{\beta^2}{k^2} e^{-2r(T-t)} \right] dt. \quad (78)$$

Solving (77) and (78) with initial conditions $E(X^*(0)) = x_0$ and $E((X^*(0))^2) = x_0^2$, gives us:

$$E(X^*(t)) = x_0 e^{rt} + \frac{c}{r}(e^{rt} - 1) + \frac{\beta^2 t}{k} e^{-r(T-t)}, \quad (79)$$

and

$$\begin{aligned} E((X^*(t))^2) &= x_0^2 e^{2rt} + \frac{c^2}{r^2}(1 - e^{2rt}) + \frac{2c}{r} \left(x_0 + \frac{c}{r} - \frac{\beta^2}{kr} e^{-rT} \right) e^{rt}(e^{rt} - 1) \\ &+ \frac{2\beta^2}{k} \left(\frac{c}{r^2} + \left(x_0 + \frac{c}{r} \right) t \right) e^{2rt-rT} - \frac{2\beta^2 c}{kr^2} (1 + rt) e^{-r(T-t)} + \frac{\beta^2 t}{k^2} (1 + \beta^2 t) e^{-2r(T-t)}. \end{aligned} \quad (80)$$

Therefore, at retirement $t = T$ we have:

$$E(X^*(T)) = \left(x_0 + \frac{c}{r} \right) e^{rT} - \frac{c}{r} + \frac{\beta^2 T}{k} = \bar{x}_0 + \frac{\beta^2 T}{k}, \quad (81)$$

and

$$E((X^*(T))^2) = e^{2rT} \left(x_0 + \frac{c}{r} \right)^2 - \frac{2c}{r} \left(x_0 + \frac{c}{r} \right) e^{rT} + \frac{c^2}{r^2} + \frac{2\beta^2 T}{k} \left(x_0 + \frac{c}{r} \right) e^{rT} + \frac{\beta^2 T}{k^2} - \frac{2c\beta^2 T}{kr} + \frac{(\beta^2 T)^2}{k^2}. \quad (82)$$

Thus, the variance of the final fund is

$$Var(X^*(T)) = E((X^*(T))^2) - E^2(X^*(T)) = \frac{\beta^2 T}{k^2}. \quad (83)$$

B.2 Logarithmic utility function

The value function is:

$$V(t, x) = \ln(b(t)) + \ln(x + a(t)), \quad (84)$$

with

$$\begin{aligned} a(t) &= \frac{c}{r}(1 - e^{-r(T-t)}), \\ b(t) &= e^{(r+\frac{\beta^2}{2})(T-t)}. \end{aligned}$$

The optimal amount invested in the risky asset is:

$$xy^*(t, x) = \frac{\beta}{\sigma} \left(x + \frac{c}{r}(1 - e^{-r(T-t)}) \right). \quad (85)$$

The evolution of the fund under optimal control $X^*(t)$ is given by:

$$dX^*(t) = \left[(r + \beta^2) X^*(t) + \left(c + \frac{c\beta^2}{r}(1 - e^{-r(T-t)}) \right) \right] dt + \beta \left(X^*(t) + \frac{c}{r}(1 - e^{-r(T-t)}) \right) dW(t). \quad (86)$$

By application of Ito's lemma to (86) the evolution of its square, $(X^*(t))^2$ is given by

$$\begin{aligned} d(X^*(t))^2 &= \left[(2r + 3\beta^2) (X^*(t))^2 + \left(2c + \frac{4c\beta^2(1-e^{-r(T-t)})}{r} \right) X^*(t) + \frac{\beta^2 c^2 (1-e^{-r(T-t)})^2}{r^2} \right] dt + \\ &+ 2\beta X^*(t) \left(X^*(t) + \frac{c(1-e^{-r(T-t)})}{r} \right) dW(t). \end{aligned} \quad (87)$$

If we take expectations on lhs and rhs in both (86) and (87) we have the following ODEs:

$$dE(X^*(t)) = \left[(r + \beta^2) E(X^*(t)) + \left(c + \frac{c\beta^2(1 - e^{-r(T-t)})}{r} \right) \right] dt, \quad (88)$$

$$\begin{aligned} dE((X^*(t))^2) &= \\ &\left[(2r + 3\beta^2) E((X^*(t))^2) + \left(2c + \frac{4c\beta^2(1-e^{-r(T-t)})}{r} \right) E(X^*(t)) + \frac{\beta^2 c^2 (1-e^{-r(T-t)})^2}{r^2} \right] dt. \end{aligned} \quad (89)$$

Solving (88) and (89) with initial conditions $E(X^*(0)) = x_0$ and $E((X^*(0))^2) = x_0^2$, gives us:

$$E(X^*(t)) = x_0 e^{At} + \frac{c}{r}(e^{At} - 1 + e^{-r(T-t)} - e^{-rT+At}), \quad (90)$$

with

$$A = r + \beta^2, \quad (91)$$

and

$$\begin{aligned} E((X^*(t))^2) &= x_0^2 e^{Kt} + \frac{c^2}{r^2} [(e^{-2r(T-t)} - e^{-2rT+Kt}) - 2(e^{-r(T-t)} - e^{-rT+Kt}) + \\ &- 2(x_0 \frac{r}{c} + 1 - e^{-rT})(e^{At} - e^{Kt} - e^{-r(T-t)+At} + e^{-rT+Kt}) + 1 - e^{Kt}], \end{aligned} \quad (92)$$

with

$$K = 2r + 3\beta^2. \quad (93)$$

Therefore, at retirement $t = T$ we have:

$$E(X^*(T)) = e^{AT} \left(x_0 + \frac{c}{r}(1 - e^{-rT}) \right) = \bar{x}_0 e^{\beta^2 T}, \quad (94)$$

and

$$E((X^*(T))^2) = e^{KT}(x_0 + \frac{c}{r}(1 - e^{-rT}))^2. \quad (95)$$

Thus, the variance of the final fund is

$$Var(X^*(T)) = (e^{KT} - e^{2AT})(x_0 + \frac{c}{r}(1 - e^{-rT}))^2 = (E(X^*(T)))^2(e^{\beta^2 T} - 1). \quad (96)$$

B.3 Power utility function

The value function is (see for instance Devolder et al. (2003)):

$$V(t, x) = b(t) \frac{(x - a(t))^\gamma}{\gamma}, \quad (97)$$

with

$$a(t) = -\frac{c}{r}(1 - e^{-r(T-t)}),$$

$$b(t) = \exp \left[\gamma \left(r + \frac{\beta^2}{2(1-\gamma)} \right) (T-t) \right].$$

The optimal amount invested in the risky asset at time t if the wealth is x is:

$$xy^*(t, x) = \frac{\beta}{\sigma(1-\gamma)} \left(x + \frac{c(1 - e^{-r(T-t)})}{r} \right). \quad (98)$$

The evolution of the fund under optimal control $X^*(t)$ is given by:

$$dX^*(t) = \left[\left(r + \frac{\beta^2}{1-\gamma} \right) X^*(t) + \left(c + \frac{c\beta^2(1 - e^{-r(T-t)})}{r(1-\gamma)} \right) \right] dt + \frac{\beta}{1-\gamma} \left(X^*(t) + \frac{c(1 - e^{-r(T-t)})}{r} \right) dW(t). \quad (99)$$

By application of Ito's lemma to (99) the evolution of its square, $(X^*(t))^2$ is given by

$$d(X^*(t))^2 = \left[\left(2r + \frac{2\beta^2}{1-\gamma} + \frac{\beta^2}{(1-\gamma)^2} \right) (X^*(t))^2 + \left(2c + \frac{2c\beta^2(1 - e^{-r(T-t)})(2-\gamma)}{r(1-\gamma)^2} \right) X^*(t) + \frac{\beta^2 c^2 (1 - e^{-r(T-t)})^2}{r^2 (1-\gamma)^2} \right] dt + \frac{2\beta X^*(t)}{1-\gamma} \left(X^*(t) + \frac{c(1 - e^{-r(T-t)})}{r} \right) dW(t). \quad (100)$$

If we take expectations on lhs and rhs in both (99) and (100) we have the following ODEs:

$$dE(X^*(t)) = \left[\left(r + \frac{\beta^2}{1-\gamma} \right) E(X^*(t)) + \left(c + \frac{c\beta^2(1 - e^{-r(T-t)})}{r(1-\gamma)} \right) \right] dt, \quad (101)$$

$$dE((X^*(t))^2) = \left[\left(2r + \frac{2\beta^2}{1-\gamma} + \frac{\beta^2}{(1-\gamma)^2} \right) E((X^*(t))^2) + \left(2c + \frac{2c\beta^2(1 - e^{-r(T-t)})(2-\gamma)}{r(1-\gamma)^2} \right) E(X^*(t)) + \frac{\beta^2 c^2 (1 - e^{-r(T-t)})^2}{r^2 (1-\gamma)^2} \right] dt. \quad (102)$$

Solving (101) and (102) with initial conditions $E(X^*(0)) = x_0$ and $E((X^*(0))^2) = x_0^2$, gives us:

$$E(X^*(t)) = x_0 e^{At} + \frac{c}{r}(e^{At} - 1 + e^{-r(T-t)} - e^{-rT+At}), \quad (103)$$

with

$$A = r + \frac{\beta^2}{1 - \gamma}, \quad (104)$$

and

$$E((X^*(t))^2) = x_0^2 e^{Kt} + \frac{c^2}{r^2} [(e^{-2r(T-t)} - e^{-2rT+Kt}) - 2(e^{-r(T-t)} - e^{-rT+Kt}) + 2(x_0 \frac{r}{c} + 1 - e^{-rT})(e^{At} - e^{Kt} - e^{-r(T-t)+At} + e^{-rT+Kt}) + 1 - e^{Kt}], \quad (105)$$

with

$$K = 2r - \frac{\beta^2(2\gamma - 3)}{(1 - \gamma)^2}. \quad (106)$$

Therefore, at retirement $t = T$ we have:

$$E(X^*(T)) = e^{AT} (x_0 + \frac{c}{r}(1 - e^{-rT})) = \bar{x}_0 e^{\frac{\beta^2 T}{1-\gamma}}, \quad (107)$$

and

$$E((X^*(T))^2) = e^{KT} (x_0 + \frac{c}{r}(1 - e^{-rT}))^2. \quad (108)$$

Thus, the variance of the final fund is

$$Var(X^*(T)) = (e^{KT} - e^{2AT}) (x_0 + \frac{c}{r}(1 - e^{-rT}))^2 = (e^{\frac{\beta^2 T}{(1-\gamma)^2}} - 1) (E(X^*(T)))^2. \quad (109)$$

It is worth noticing that, apart from the value function, the results for the logarithmic utility can be found by setting $\gamma = 0$ in the power case, that is an expected result.

C Proof of Theorem 3

We will follow the procedure as in (2a). For notational convenience, we will write $X^*(T)$ rather than $X_U^*(T)$.

i) For the exponential utility function, the final fund under optimal control has the following mean (see (81)):

$$E(X^*(T)) = \bar{x}_0 + \frac{\beta^2 T}{k}, \quad (110)$$

and the following variance (see (83)):

$$Var(X^*(T)) = \frac{\beta^2 T}{k^2}. \quad (111)$$

The expected final funds given in (43) and (110) are equal if and only if

$$\frac{\beta^2 T}{k} = \frac{e^{\beta^2 T} - 1}{2\alpha}. \quad (112)$$

We need to prove that, under (112), the variance (111) is higher than the variance in the mean-variance efficient case, (44), i.e. we have to prove that

$$\frac{\beta^2 T}{k^2} - \frac{e^{\beta^2 T} - 1}{4\alpha^2} > 0. \quad (113)$$

Using (112) it is easy to see that

$$\frac{\beta^2 T}{k^2} - \frac{e^{\beta^2 T} - 1}{4\alpha^2} = \frac{e^{\beta^2 T} - 1}{2\alpha} \left(\frac{1}{k} - \frac{1}{2\alpha} \right) > 0 \Leftrightarrow \frac{k}{2\alpha} < 1. \quad (114)$$

The last inequality is true, since, due to (112),

$$\frac{k}{2\alpha} = \frac{\beta^2 T}{e^{\beta^2 T} - 1} < 1 \quad \text{for} \quad \beta^2 T \neq 0. \quad (115)$$

ii) For the logarithmic utility function, the final fund under optimal control has the following mean (see (94)):

$$E(X^*(T)) = \bar{x}_0 e^{\beta^2 T}, \quad (116)$$

and the following variance (see (96)):

$$Var(X^*(T)) = (E(X^*(T)))^2 (e^{\beta^2 T} - 1). \quad (117)$$

The expected final funds given in (116) and (43) are equal if and only if

$$e^{\beta^2 T} - 1 = \frac{e^{\beta^2 T} - 1}{2\alpha\bar{x}_0}, \quad (118)$$

which happens if and only if

$$\alpha = \frac{1}{2\bar{x}_0}. \quad (119)$$

Proving that, under (119), the variance (117) is higher than the variance in the mean-variance efficient case, (44), is straightforward. In fact:

$$(\bar{x}_0 e^{\beta^2 T})^2 (e^{\beta^2 T} - 1) - \frac{e^{\beta^2 T} - 1}{4\alpha^2} = \bar{x}_0^2 e^{2\beta^2 T} (e^{\beta^2 T} - 1) - \bar{x}_0^2 (e^{\beta^2 T} - 1) = \bar{x}_0^2 (e^{\beta^2 T} - 1)^2 (e^{\beta^2 T} + 1) > 0. \quad (120)$$

iii) For the power utility function, the final fund under optimal control has the following mean (see (107)):

$$E(X^*(T)) = \bar{x}_0 e^{\frac{\beta^2 T}{1-\gamma}}, \quad (121)$$

and the following variance (see (109)):

$$Var(X^*(T)) = (E(X^*(T)))^2 (e^{\frac{\beta^2 T}{(1-\gamma)^2}} - 1). \quad (122)$$

The expected final funds given in (121) and (43) are equal if and only if

$$e^{\frac{\beta^2 T}{1-\gamma}} - 1 = \frac{e^{\beta^2 T} - 1}{2\alpha\bar{x}_0}. \quad (123)$$

We intend to prove that, under (123), the variance (122) is higher than the variance in the mean-variance efficient case, (44), i.e. we have to prove that

$$(\bar{x}_0 e^{\frac{\beta^2 T}{1-\gamma}})^2 (e^{\frac{\beta^2 T}{(1-\gamma)^2}} - 1) - \frac{e^{\beta^2 T} - 1}{4\alpha^2} > 0. \quad (124)$$

For the special case $0 < \gamma < 1$ and $e^{\beta^2 T} - 1 > 1$ it is possible to prove the claim (124) in a straightforward way. However, here we present the complete proof that covers all possible values of $\gamma \in (-\infty, 1)$ and $e^{\beta^2 T} \in (1, +\infty)$.

It is easy to see that, under (123) the claim (124) is true if

$$e^{\frac{2\beta^2 T}{(1-\gamma)}} (e^{\frac{\beta^2 T}{(1-\gamma)^2}} - 1)(e^{\beta^2 T} - 1) - (e^{\frac{\beta^2 T}{1-\gamma}} - 1)^2 > 0. \quad (125)$$

With the change of variable

$$b := e^{\beta^2 T} \quad a := \frac{1}{1-\gamma}, \quad (126)$$

our claim (125) is now

$$b^{2a}(b^{a^2} - 1)(b - 1) - (b^a - 1)^2 = b^{a^2+2a+1} - b^{a^2+2a} - b^{2a+1} + 2b^a - 1 > 0 \quad (127)$$

for all $(a, b) \in (0, +\infty) \times (1, +\infty)$. The claim in (127) is equivalent to

$$b^{a^2+2a+1} + 2b^a > b^{a^2+2a} + b^{2a+1} + 1. \quad (128)$$

For fixed $b \in (1, +\infty)$, this is equivalent to show that

$$f_b(a) > g_b(a) \quad \forall a \in (0, +\infty) \quad (129)$$

with

$$f_b(a) := b^{a^2+2a+1} + 2b^a, \quad (130)$$

and

$$g_b(a) := b^{a^2+2a} + b^{2a+1} + 1. \quad (131)$$

We have

$$f'_b(a) = (2b^a + (2a + 2)b^{a^2+2a+1}) \log b \quad (132)$$

$$f''_b(a) = (2b^a + (2a + 2)^2 b^{a^2+2a+1})(\log b)^2 + 2b^{a^2+2a+1} \log b \quad (133)$$

and

$$g'_b(a) = (2b^{2a+1} + (2a + 2)b^{a^2+2a}) \log b \quad (134)$$

$$g''_b(a) = (4b^{2a+1} + (2a + 2)^2 b^{a^2+2a})(\log b)^2 + 2b^{a^2+2a} \log b. \quad (135)$$

Then

$$\lim_{a \rightarrow 0^+} f_b(a) = \lim_{a \rightarrow 0^+} g_b(a) = 2 + b, \quad (136)$$

$$\lim_{a \rightarrow 0^+} f'_b(a) = \lim_{a \rightarrow 0^+} g'_b(a) = (2 + 2b) \log b. \quad (137)$$

However,

$$\lim_{a \rightarrow 0^+} f''_b(a) = (4b + 2)(\log b)^2 + 2b \log b \quad (138)$$

$$\lim_{a \rightarrow 0^+} g''_b(a) = (4 + 4b)(\log b)^2 + 2 \log b \quad (139)$$

so that, since $b > 1$, we have:

$$f''_b(0) - g''_b(0) = 2 \log b(b - 1 - \log b) > 0. \quad (140)$$

As a consequence, if we expand $f_b(a) - g_b(a)$ close to 0^+ with the Taylor series, we have that for $a \rightarrow 0^+$

$$f_b(a) - g_b(a) = \log b(b - 1 + \log b)a^2 + o(a^2) \quad (141)$$

and we conclude that

$$f_b(a) > g_b(a) \quad (142)$$

for $a \rightarrow 0^+$.

Since $f'_b(0) = g'_b(0)$ and $f''_b(0) > g''_b(0)$, if we show that $f''_b(a) > g''_b(a)$ for all $a \in (0, +\infty)$ the claim (129) is proven. We have:

$$\begin{aligned} f''_b(a) - g''_b(a) &= (\log b)^2(2b^a + (2a + 2)^2b^{a^2+2a+1} - 4b^{2a+1} + (2a + 2)^2b^{a^2+2a}) + 2\log b(b^{a^2+2a+1} - b^{a^2+2a}) \\ &= 2\log b[(2(a + 1)^2(b^{a^2+2a+1} - b^{a^2+2a}) + b^a - 2b^{2a+1})\log b + (b^{a^2+2a+1} - b^{a^2+2a})]. \end{aligned} \quad (143)$$

We have

$$f''_b(a) > g''_b(a) \quad (144)$$

if and only if

$$(2(a + 1)^2(b^{a^2+2a+1} - b^{a^2+2a}) + b^a - 2b^{2a+1})\log b + (b^{a^2+2a+1} - b^{a^2+2a}) > 0 \quad (145)$$

that is true if and only if

$$(b^{a^2+2a+1} - b^{a^2+2a})(1 + 2\log b(a + 1)^2) > (2b^{2a+1} - b^a)\log b \quad (146)$$

which is equivalent to

$$h(a) > k(a) \quad (147)$$

for $a \in (0, +\infty)$ with

$$h(a) := (b^{a^2+2a+1} - b^{a^2+2a})(1 + 2\log b(a + 1)^2) \quad (148)$$

and

$$k(a) := (2b^{2a+1} - b^a)\log b. \quad (149)$$

It is easy to see that

$$h(0) - k(0) = b - 1 - \log b > 0. \quad (150)$$

It is also possible to show that $h'(a) > k'(a)$. In fact,

$$h'(a) = (a + 1)\log b(b^{a^2+2a+1} - b^{a^2+2a})(6 + 4\log b(a + 1)^2) \quad (151)$$

and

$$k'(a) = (\log b)^2(4b^{2a+1} - b^a). \quad (152)$$

Therefore, using the fact that $b - 1 > \log b$, we have

$$\begin{aligned} h'(a) - k'(a) &= (a + 1)\log b(b^{a^2+2a+1} - b^{a^2+2a})(6 + 4\log b(a + 1)^2) - (\log b)^2(4b^{2a+1} - b^a) \\ &= 4(\log b)^2(b^{a^2+2a+1} - b^{2a+1}) + a\log b(6 + 4\log b(a + 1)^2)(b^{a^2+2a+1} - b^{a^2+2a}) + \\ &\quad + \log b(6 + 4\log b(a^2 + 2a))b^{a^2+2a}(b - 1) - 4(\log b)^2b^{a^2+2a} + (\log b)^2b^a \\ &> 4(\log b)^2(b^{a^2+2a+1} - b^{2a+1}) + a\log b(6 + 4\log b(a + 1)^2)(b^{a^2+2a+1} - b^{a^2+2a}) + \\ &\quad + 2(\log b)^2b^{a^2+2a} + 4(\log b)^3(a^2 + 2a)b^{a^2+2a} + (\log b)^2b^a > 0. \end{aligned} \quad (153)$$

Since $h(0) > k(0)$ and $h'(a) > k'(a)$ for all $a > 0$, (147) holds. This in turn implies (144), that implies (127). \square

D Proof of Theorem 5

i) We first set

$$E(\bar{X}(T)) = E(X^*(T)). \quad (154)$$

From (60) we have

$$e^{\beta^2 T} E(X^*(T)) = \bar{x}_0 + F(e^{\beta^2 T} - 1). \quad (155)$$

Then, applying (21) and (154), yields

$$e^{\beta^2 T} E(\bar{X}(T)) = E(\bar{X}(T)) - \frac{e^{\beta^2 T} - 1}{2\alpha} + F(e^{\beta^2 T} - 1). \quad (156)$$

Collecting terms and dividing by $e^{\beta^2 T} - 1 > 0$, we have

$$E(\bar{X}(T)) = F - \frac{1}{2\alpha}. \quad (157)$$

We now have:

$$y_{tb}(t, x) = -\frac{\lambda - r}{\sigma^2 x} (x - G(t)) \quad (158)$$

$$= -\frac{\lambda - r}{\sigma^2 x} \left\{ x - \left[F e^{-r(T-t)} - \frac{c}{r} (1 - e^{-r(T-t)}) \right] \right\} \quad (159)$$

$$= -\frac{\lambda - r}{\sigma^2 x} \left\{ x - \left[\left(F - \frac{1}{2\alpha} \right) e^{-r(T-t)} - \frac{c}{r} (1 - e^{-r(T-t)}) + \frac{e^{-r(T-t)}}{2\alpha} \right] \right\}$$

$$= \bar{y}(t, x),$$

where in the last equality we have used (157) and (23). It is then clear that, since $y_{tb}(t, x)$ is a particular case of mean-variance investment strategy, it must lead to an optimal portfolio that is mean-variance efficient.

ii) Consider a point $(Var(\bar{X}(T)), E(\bar{X}(T)))$ on the efficient frontier. Using (21) it is possible to find the corresponding $\bar{\alpha}$ which in turn defines the target via (157):

$$\bar{F} = E(\bar{X}(T)) + \frac{1}{2\bar{\alpha}}. \quad (160)$$

It is then obvious that the point $(Var(\bar{X}(T)), E(\bar{X}(T)))$ chosen on the efficient frontier can be found by solving the target-based optimization problem with target equal to \bar{F} . \square